

REGULAR REPRESENTATIONS OF THE QUANTUM GROUPS AT ROOTS OF UNITY

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ABSTRACT. We study the bimodule structure of the quantum function algebra at roots of 1 and prove that it admits an increasing filtration with factors isomorphic to the tensor products of the dual of Weyl modules $V_\lambda^* \otimes V_{-\omega_0\lambda}^*$. As an application we compute the 0-th Hochschild cohomology of the function algebra at roots of 1.

1. INTRODUCTION

Let \mathfrak{g} be a simple complex Lie algebra, and let \mathbf{U} be the quantized enveloping algebra of \mathfrak{g} over $\mathbb{Q}(v)$, v an indeterminate. Let \mathbf{O} be the linear span of matrix coefficients of finite-dimensional \mathbf{U} -modules (see [D]), then there is a perfect Hopf algebra pairing between \mathbf{U} and \mathbf{O} . Moreover, as a $\mathbf{U} \otimes \mathbf{U}$ -module, \mathbf{O} has a classical Peter-Weyl type of decomposition. Following [L3], set $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ and let U denote Lusztig's \mathcal{A} -form of \mathbf{U} generated by the divided powers. Set $\mathcal{A}_0 = \mathbb{Q}[v, v^{-1}]$ and $U_{\mathcal{A}_0} = U \otimes_{\mathcal{A}} \mathcal{A}_0$. Let $U_{\mathcal{A}_0}^*$ be the set of all \mathcal{A}_0 -linear maps $U_{\mathcal{A}_0} \rightarrow \mathcal{A}_0$, and set $O = \mathbf{O} \cap U_{\mathcal{A}_0}^*$. Then O is a Hopf algebra over \mathcal{A}_0 and the inclusion $O \subset \mathbf{O}$ induces an isomorphism of $\mathbb{Q}(v)$ -algebras: $O \otimes_{\mathcal{A}_0} \mathbb{Q}(v) \cong \mathbf{O}$. Now let $q \in \mathbb{C}$ be a primitive ℓ -th root of unity; set $U_q = U_{\mathcal{A}_0} \otimes_{\mathcal{A}_0} \mathbb{Q}(q)$ and $O_q = O \otimes_{\mathcal{A}_0} \mathbb{Q}(q)$, where $\mathbb{Q}(q)$ is made into an \mathcal{A}_0 -algebra by specializing v to q . There is a Hopf algebra pairing between U_q and O_q , thus O_q admits the structure of a $U_q \times U_q$ -module. The main goal of the present paper is to investigate this bimodule structure.

Our motivation comes from a family of vertex operator algebras associated to the modified regular representations of the affine Lie algebra $\hat{\mathfrak{g}}$ (see [Z1] and references therein). Each one of these vertex operator algebras admits two commuting actions of $\hat{\mathfrak{g}}$ in dual levels. When the dual central charges are generic, it decomposes into summands corresponding to the dominant weights of \mathfrak{g} . However when the dual central charges are rational, the module structure is less well understood, and it should be closely related to the regular representation of the corresponding quantum group at a root of unity, by the equivalence of tensor categories established by Kazhdan and Lusztig between representations of the affine Lie algebra and representations of the quantum group (see [KL1-4]).

For simplicity, we assume $\mathfrak{g} = \mathfrak{sl}_n$ is of type A , and $\ell \geq n$ is odd. The quantum coordinate algebra \mathbf{O} of \mathfrak{sl}_n can be described by generators and relations. Let V be the quantization of the n -dimensional natural representation of \mathfrak{sl}_n , then \mathbf{O} is generated by the matrix coefficients $X_{ij}, 1 \leq i, j \leq n$ of V subject to a list of relations (see [D], [T], [APW]). In Section 2, we will show that O , as an \mathcal{A}_0 -subalgebra of \mathbf{O} , is generated by X_{ij} 's over \mathcal{A}_0 , which generalizes Proposition 1.3 of [CL]. A similar result was obtained in the Appendix of [APW] by P. Polo, using some local ring as the basic ring. Another \mathcal{A}_0 -form of \mathbf{U} , denoted by $\Gamma(\mathfrak{g})$ and slightly different from $U_{\mathcal{A}_0}$, was introduced in

[CL]. The dual of $\Gamma(\mathfrak{g})$, appropriately defined, coincides with O . Specializing v to q , a primitive ℓ -th root of 1, we get a perfect pairing between $\Gamma_q(\mathfrak{g}) = \Gamma(\mathfrak{g}) \otimes_{\mathcal{A}_0} \mathbb{Q}(q)$ and $O_q = O \otimes_{\mathcal{A}_0} \mathbb{Q}(q)$, hence it induces an embedding $O_q \hookrightarrow \Gamma_q(\mathfrak{g})^*$ ([CL, Lemma 6.1]). When $U_q = U_{\mathcal{A}_0} \otimes_{\mathcal{A}_0} \mathbb{Q}(q)$ and O_q are concerned, we still have the embedding $O_q \hookrightarrow U_q^*$, though the pairing between U_q and O_q is in general degenerate on the U_q -side.

Consider the non-semisimple category \mathcal{C}_f of finite dimensional U_q -modules (of type 1). The quantum function algebra O_q can be realized as the linear span of matrix coefficients of modules from \mathcal{C}_f . One of our main results is that O_q , as a $U_q \times U_q$ -module, admits an increasing filtration with factors isomorphic to the tensor products of the dual of Weyl modules $V_\lambda^* \otimes V_{-\omega_0\lambda}^*$. It can be regarded as a generalization of the Peter-Weyl type of decomposition for O_q , q not a root of unity. Similar results for the regular representation of the affine Lie algebra in a rational level provide a proof of the conjecture, stated at the end of [Z1], about the bimodule structure of a family of vertex operator algebras in rational levels (see [Z2]). As an application of this increasing filtration of O_q , we show that the cocommutative elements of O_q are linear combinations of the “traces” of modules from \mathcal{C}_f , moreover as an algebra, it is isomorphic to the Grothendieck ring of \mathcal{C}_f extended to the field $\mathbb{Q}(q)$.

The paper is organized as follows: Section 2 gives an overview of the quantized enveloping algebra \mathbf{U} , the quantum coordinate algebra \mathbf{O} , Lusztig’s \mathcal{A} -form U of \mathbf{U} , the corresponding \mathcal{A}_0 -form O of \mathbf{O} , and their specializations U_q, O_q at a root of unity. In particular we describe O for type A (Proposition 2.1), and identify O_q with the matrix coefficients of finite dimensional U_q -modules (Proposition 2.4). In Section 3, we describe an increasing filtration of O_q using tilting modules (Theorem 3.3), and compute the 0-th Hochschild cohomology of O_q as a coalgebra (Proposition 3.5). We treat the \mathfrak{sl}_2 case more thoroughly in Section 4 and are able to obtain more explicit results (Theorem 4.6).

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2. THE GENERAL SETTING

Let $(a_{ij})_{1 \leq i, j \leq n-1}$ be the Cartan matrix of a simply-laced simple Lie algebra \mathfrak{g} . The quantized enveloping algebra \mathbf{U} is the $\mathbb{Q}(v)$ -algebra defined by the generators E_i, F_i, K_i, K_i^{-1} ($1 \leq i \leq n-1$) and the relations

$$\begin{aligned} K_i K_i^{-1} &= K_i^{-1} K_i = 1, & K_i K_j &= K_j K_i, \\ K_i E_j &= v^{a_{ij}} E_j K_i, & K_i F_j &= v^{-a_{ij}} F_j K_i, \\ E_i F_j - F_j E_i &= \delta_{ij} \frac{K_i - K_i^{-1}}{v - v^{-1}}, \\ E_i E_j &= E_j E_i, & F_i F_j &= F_j F_i, & \text{if } a_{ij} = 0, \\ E_i^2 E_j - (v + v^{-1}) E_i E_j E_i + E_j E_i^2 &= 0, & \text{if } a_{ij} = -1, \\ F_i^2 F_j - (v + v^{-1}) F_i F_j F_i + F_j F_i^2 &= 0, & \text{if } a_{ij} = -1. \end{aligned}$$

\mathbf{U} is a Hopf algebra over $\mathbb{Q}(v)$ with comultiplication Δ , counit ε and antipode S defined by

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \Delta(K_i) = K_i \otimes K_i,$$

$$\begin{aligned}\varepsilon(E_i) &= \varepsilon(F_i) = 0, & \varepsilon(K_i) &= 1, \\ S(E_i) &= -K_i^{-1}E_i, & S(F_i) &= -F_iK_i, & S(K_i) &= K_i^{-1}.\end{aligned}$$

Following [L3, Sect 7], let \mathcal{F} be the set of all two-sided ideals I in \mathbf{U} such that I has finite codimension and there exists some $r \in \mathbb{N}$ such that for any i we have $\prod_{h=-r}^r (K_i - v^h) \in I$. Let \mathbf{O} be the set of all $\mathbb{Q}(v)$ -linear maps $f : \mathbf{U} \rightarrow \mathbb{Q}(v)$ such that $f|_I = 0$ for some $I \in \mathcal{F}$. We call \mathbf{O} the quantum coordinate (or function) algebra, which is equivalent to the linear span of matrix coefficients of finite dimensional \mathbf{U} -modules with a weight decomposition. Moreover \mathbf{O} is a Hopf algebra over $\mathbb{Q}(v)$, and there exists a perfect Hopf algebra pairing $\mathbf{U} \times \mathbf{O} \rightarrow \mathbb{Q}(v)$. Since all finite dimensional \mathbf{U} -modules are completely reducible, \mathbf{O} has a classical Peter-Weyl type of decomposition as a $\mathbf{U} \times \mathbf{U}$ -module.

For type A , we can describe \mathbf{O} by generators and relations. Set $\mathfrak{g} = \mathfrak{sl}_n$, and let $(a_{ij})_{1 \leq i, j \leq n-1}$ be the Cartan matrix with $a_{ij} = -1$ if $|i - j| = 1$; 2 if $i = j$; 0 otherwise. Let $\alpha_1, \dots, \alpha_{n-1}$ be the simple roots of \mathfrak{sl}_n associated to $(a_{ij})_{1 \leq i, j \leq n-1}$, and let $\omega_1, \dots, \omega_{n-1}$ be the corresponding fundamental weights. Let V denote the quantization of the n -dimensional natural representation of \mathfrak{sl}_n with highest weight ω_1 . Fix a highest weight vector $x_1 \in V$, and set $x_{i+1} = F_i x_i$ for all $1 \leq i \leq n-1$. Then x_i has weight $\omega_i - \omega_{i-1}$ (with the convention $\omega_0 = \omega_n = 0$), and $\{x_1, \dots, x_n\}$ form a $\mathbb{Q}(v)$ -basis of V with $E_i x_{i+1} = x_i$. Let $\{\delta_1, \dots, \delta_n\}$ be the dual basis in V^* , and define $X_{ij} \in \mathbf{U}^*$ by $X_{ij}(u) = \delta_i(u \cdot x_j)$ for any $u \in \mathbf{U}$. These functionals X_{ij} belong to \mathbf{O} and they satisfy the following relations:

$$\begin{aligned}X_{il}X_{jl} - vX_{jl}X_{il} &= 0 & \text{for all } l, i < j \\ X_{li}X_{lj} - vX_{lj}X_{li} &= 0 & \text{for all } l, i < j \\ X_{li}X_{mj} - X_{mj}X_{li} &= 0 & \text{if } l < m \text{ and } i > j \\ X_{li}X_{mj} - X_{mj}X_{li} - (v - v^{-1})X_{lj}X_{mi} &= 0 & \text{if } l < m \text{ and } i < j \\ \sum_{\sigma \in S_n} (-v)^{l(\sigma)} X_{\sigma(1)1} \cdots X_{\sigma(n)n} &= 1\end{aligned}$$

where $l(\sigma)$ denotes the length of the permutation. In fact \mathbf{O} is generated by X_{ij} , $1 \leq i, j \leq n$ subject to the above relations (see [D], [T]).

Set $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. Given $n \in \mathbb{Z}$, $m \in \mathbb{N}$, we define $[n] = \frac{v^n - v^{-n}}{v - v^{-1}} \in \mathcal{A}$, $[m]! = [m][m-1] \cdots [1]$ and $\begin{bmatrix} n \\ m \end{bmatrix} = \prod_{j=1}^m \frac{v^{n-j+1} - v^{-n+j-1}}{v^j - v^{-j}} \in \mathcal{A}$. Following [L1, L3], let U be the \mathcal{A} -subalgebra of \mathbf{U} generated by the elements $E_i^{(N)} = E_i^N / [N]!$, $F_i^{(N)} = F_i^N / [N]!$, K_i , K_i^{-1} ($1 \leq i \leq n-1$, $N \geq 0$). Then U is a free \mathcal{A} -module and is itself a Hopf algebra over \mathcal{A} in a natural way. Let U^+ , U^- , U^0 be the \mathcal{A} -subalgebras of U generated by the elements $E_i^{(N)}$; $F_i^{(N)}$; $K_i^{\pm 1}$, $\begin{bmatrix} K_i; c \\ t \end{bmatrix}$, where

$$\begin{bmatrix} K_i; c \\ t \end{bmatrix} = \prod_{s=1}^t \frac{K_i v^{c-s+1} - K_i^{-1} v^{-c+s-1}}{v^s - v^{-s}},$$

then multiplication induces an isomorphism of \mathcal{A} -modules: $U^- \otimes U^0 \otimes U^+ \cong U$.

We follow [L1] to construct an \mathcal{A} -basis of U . Let s_i , $1 \leq i \leq n-1$ be the simple reflections in the Weyl group $\mathcal{W} = S_n$ of \mathfrak{sl}_n . Let R , R^+ denote the root system and positive roots. Set $\alpha_{ij} = s_j s_{j-1} \cdots s_{i+1} \alpha_i = \sum_{k=i}^j \alpha_k \in R^+$ for any $1 \leq i < j \leq n-1$,

and consider the following total order on R^+ : $\alpha_{n-1} < \alpha_{n-2,n-1} < \cdots < \alpha_{1,n-1} < \alpha_{n-2} < \alpha_{n-3,n-2} < \cdots < \alpha_{1,n-2} < \cdots < \alpha_2 < \alpha_{12} < \alpha_1$. Let $\Omega : \mathbf{U} \rightarrow \mathbf{U}^{\text{opp}}$ be the \mathbb{Q} -algebra isomorphism and $T_i : \mathbf{U} \rightarrow \mathbf{U}$, $1 \leq i \leq n-1$ be the $\mathbb{Q}(v)$ -algebra isomorphism defined in [L1, Sect 1]. Set $E_{ij} = T_j T_{j-1} \cdots T_{i+1} E_i$, and define for any $\phi, \phi' \in \mathbb{N}^{R^+}$,

$$E^\phi = \prod_{\beta \in R^+} E_\beta^{(\phi(\beta))}, \quad F^{\phi'} = \Omega(E^{\phi'}),$$

where $E_{\alpha_i} = E_i$, $E_{\alpha_{ij}} = E_{ij}$, $E_\beta^{(N)} = E_\beta^N / [N]!$ and the factors in E^ϕ are written in the given order of R^+ . Then the elements $E^\phi; F^{\phi'}; \prod_{i=1}^{n-1} K_i^{\delta_i} \begin{bmatrix} K_i; 0 \\ t_i \end{bmatrix}$, $t_i \geq 0, \delta_i = 0$ or 1 form an \mathcal{A} -basis of $U^+; U^-; U^0$ respectively. Hence the elements $F^{\phi'} K E^\phi$, with K in the above \mathcal{A} -basis of U^0 , form an \mathcal{A} -basis of U . They also form a $\mathbb{Q}(v)$ -basis of \mathbf{U} , hence we have an isomorphism of $\mathbb{Q}(v)$ -algebras: $U \otimes_{\mathcal{A}} \mathbb{Q}(v) \cong \mathbf{U}$.

Again following [L3, Sect 7], set $\mathcal{A}_0 = \mathbb{Q}[v, v^{-1}]$ and $U_{\mathcal{A}_0} = U \otimes_{\mathcal{A}} \mathcal{A}_0$. Let $U_{\mathcal{A}_0}^*$ be the set of all \mathcal{A}_0 -linear maps $U_{\mathcal{A}_0} \rightarrow \mathcal{A}_0$ and let $O = \mathbf{O} \cap U_{\mathcal{A}_0}^*$. Then O is a Hopf algebra over \mathcal{A}_0 , and the inclusion $O \hookrightarrow \mathbf{O}$ induces an isomorphism of Hopf $\mathbb{Q}(v)$ -algebras: $O \otimes_{\mathcal{A}_0} \mathbb{Q}(v) \cong \mathbf{O}$. Let \mathcal{M} be a $U_{\mathcal{A}_0}$ -module, which is a free \mathcal{A}_0 -module of finite rank with a basis in which the operators $K_i, \begin{bmatrix} K_i; 0 \\ t_i \end{bmatrix}$ act by diagonal matrices

with eigenvalues $v^m, \begin{bmatrix} m \\ t \end{bmatrix}$. For any $m \in \mathcal{M}$ and $\xi \in \text{Hom}_{\mathcal{A}_0}(\mathcal{M}, \mathcal{A}_0)$, the matrix coefficient $c_{m,\xi} : u \rightarrow \xi(u \cdot m)$, an element of $U_{\mathcal{A}_0}^*$, belongs to O . Moreover O is exactly the \mathcal{A}_0 -submodule of $U_{\mathcal{A}_0}^*$ spanned by the matrix coefficients $c_{m,\xi}$ for various \mathcal{M}, m, ξ as above.

Another integral form of \mathbf{U} , denoted by $\Gamma(\mathfrak{g})$, was introduced in [CL]. By definition $\Gamma(\mathfrak{g})$ is the \mathcal{A}_0 -subalgebra of \mathbf{U} generated by $E_i^{(N)}, F_i^{(N)}, K_i^{\pm 1}, \begin{pmatrix} K_i; c \\ t \end{pmatrix}$, where

$$\begin{pmatrix} K_i; c \\ t \end{pmatrix} = \prod_{s=1}^t \frac{K_i v^{c-s+1} - 1}{v^s - 1}. \quad \text{This algebra is larger than } U_{\mathcal{A}_0} \text{ because its Cartan}$$

part $\Gamma(\mathfrak{t})$ is larger than that of $U_{\mathcal{A}_0}$. The elements $\prod_{i=1}^{n-1} \begin{pmatrix} K_i; 0 \\ t_i \end{pmatrix} K_i^{-[t_i/2]}$ ($t_i \geq 0$) form an \mathcal{A}_0 -basis of $\Gamma(\mathfrak{t})$, where the Gauss symbol $[x]$ denotes the largest integer that is not greater than x . Let \mathcal{C} be the full subcategory of $\Gamma(\mathfrak{g})$ -modules, which is free of finite rank as an \mathcal{A}_0 -module and has a basis in which the operators $K_i, \begin{pmatrix} K_i; 0 \\ t \end{pmatrix}$ act

by diagonal matrices with eigenvalues $v^m, \begin{pmatrix} m \\ t \end{pmatrix}$, where $\begin{pmatrix} m \\ t \end{pmatrix} = \prod_{s=1}^t \frac{v^{m-s+1} - 1}{v^s - 1}$.

Then the dual of $\Gamma(\mathfrak{g})$, defined to be the linear span of matrix coefficients of modules from \mathcal{C} , coincides with O ([CL, Remark 4.1]).

Recall that for $\mathfrak{g} = \mathfrak{sl}_n$, the quantum coordinate algebra \mathbf{O} is generated by $X_{ij}, 1 \leq i, j \leq n$ over $\mathbb{Q}(v)$ subject to some relations. We want to show that its subalgebra O is generated by $X_{ij}, 1 \leq i, j \leq n$ over \mathcal{A}_0 . It generalizes Proposition 1.3 of [CL], and the proof written below is pure calculation.

Proposition 2.1. *The \mathcal{A}_0 -subalgebra O of \mathbf{O} is generated by $X_{ij}, 1 \leq i, j \leq n$.*

Proof. Let Ξ be the set of all matrices $M = (r_{ij})_{1 \leq i, j \leq n}$ such that $r_{ij} \in \mathbb{N}$ and at least one of r_{11}, \dots, r_{nn} is zero. Fix a total order on $\{1, \dots, n\}^2$, and set $X^M = \prod_{ij} X_{ij}^{r_{ij}}$ for any $M \in \Xi$. Then $\{X^M, M \in \Xi\}$ form a $\mathbb{Q}(v)$ -basis of \mathbf{O} . Any element $f \in O$ can be represented uniquely as $f = \sum_{M \in \Xi} \gamma_M X^M$ with $\gamma_M \in \mathbb{Q}(v)$. Since $X_{ij} \in O$ for all $1 \leq i, j \leq n$, it suffices to prove that one of the nonzero coefficients γ_M belongs to \mathcal{A}_0 . Define a set of nonnegative integers inductively as follows:

$$\begin{aligned} s_{n1} &= \min\{r_{n1} | \gamma_{M=(r_{ij})} \neq 0\} \\ s_{n-1,1} &= \min\{r_{n-1,1} | \gamma_{M=(r_{ij})} \neq 0, r_{n1} = s_{n1}\} \\ &\dots \\ s_{21} &= \min\{r_{21} | \gamma_{M=(r_{ij})} \neq 0, r_{i1} = s_{i1}, 2 < i \leq n\} \\ s_{n2} &= \min\{r_{n2} | \gamma_{M=(r_{ij})} \neq 0, r_{i1} = s_{i1}, 1 < i \leq n\} \\ &\dots \\ s_{32} &= \min\{r_{32} | \gamma_{M=(r_{ij})} \neq 0, r_{j2} = s_{j2}, r_{i1} = s_{i1}, 3 < j \leq n, 1 < i \leq n\} \\ &\dots \\ s_{n,n-1} &= \min\{r_{n,n-1} | \gamma_{M=(r_{ij})} \neq 0, r_{ij} = s_{ij}, 1 \leq j < n-1, j < i \leq n\}. \end{aligned}$$

Define $\phi \in \mathbb{N}^{R^+}$; $\alpha_i \mapsto s_{i+1,i}, \alpha_{ij} \mapsto s_{j+1,i}$, then $f(F^\phi KE) = \sum_{M \in \Lambda} \gamma_M X^M (F^\phi KE)$ for any $K \in U^0, E \in U^+$, where $\Lambda = \{M = (r_{ij}) \in \Xi | \gamma_M \neq 0, r_{ij} = s_{ij}, 1 \leq j < i \leq n\}$. Similarly define another set of nonnegative integers:

$$\begin{aligned} s_{1n} &= \min\{r_{1n} | M = (r_{ij}) \in \Lambda\} \\ &\dots \\ s_{12} &= \min\{r_{12} | M = (r_{ij}) \in \Lambda, r_{1i} = s_{1i}, 2 < i \leq n\} \\ s_{2n} &= \min\{r_{2n} | M = (r_{ij}) \in \Lambda, r_{1i} = s_{1i}, 1 < i \leq n\} \\ &\dots \\ s_{23} &= \min\{r_{23} | M = (r_{ij}) \in \Lambda, r_{2j} = s_{2j}, r_{1i} = s_{1i}, 3 < j \leq n, 1 < i \leq n\} \\ &\dots \\ s_{n-1,n} &= \min\{r_{n-1,n} | M = (r_{ij}) \in \Lambda, r_{ij} = s_{ij}, 1 \leq i < n-1, i < j \leq n\}. \end{aligned}$$

Define $\psi \in \mathbb{N}^{R^+}$; $\alpha_i \mapsto s_{i,i+1}, \alpha_{ij} \mapsto s_{i,j+1}$, then $f(F^\psi KE) = \sum_{M \in \Upsilon} \gamma_M X^M (F^\psi KE)$ for any $K \in U^0$, where $\Upsilon = \{M = (r_{ij}) \in \Lambda | r_{ij} = s_{ij}, 1 \leq i < j \leq n\}$. In other words, Υ is the collection of matrices $M \in \Xi$ whose non-diagonal entries are s_{ij} 's and the coefficient of X^M in $f \in O$ is nonzero, moreover $f(F^\psi KE) = \sum_{M \in \Upsilon} \gamma_M v^{n_M} \chi_{\lambda + \mu_M}(K)$ for some $n_M \in \mathbb{Z}$, where $\lambda = \sum_{i > j} s_{ij}(\omega_j - \omega_{j-1}) + \sum_{i < j} s_{ij}(\omega_i - \omega_{i-1})$ and $\mu_M = \sum_{i=1}^{n-1} (r_{ii} - r_{i+1,i+1})\omega_i$ for $M = (r_{ij}) \in \Upsilon$. Here the character $\chi_\nu : U^0 \rightarrow \mathcal{A}$ associated to a weight $\nu = (\nu_i), \nu_i = \langle \nu, \alpha_i^\vee \rangle$ is defined by $\chi_\nu(K_i^{\pm 1}) = v^{\pm \nu_i}, \chi_\nu\left(\begin{bmatrix} K_i & c \\ & t \end{bmatrix}\right) = \begin{bmatrix} \nu_i + c \\ t \end{bmatrix}$. Note that $\mu_M = \mu_{M'}$ if and only if $M = M'$. Assume the diagonal entries of M and M' are (r_{11}, \dots, r_{nn}) and $(r'_{11}, \dots, r'_{nn})$ respectively, then $\mu_M = \mu_{M'}$ iff $r_{ii} - r_{i+1,i+1} = r'_{ii} - r'_{i+1,i+1}$, which is equivalent to $r_{ii} - r'_{ii} = c, 1 \leq i \leq n$ for some constant c . Since $r_{ii}, r'_{ii} \in \mathbb{N}$ and at least one from each group is zero, c must be zero. Now that the characters $\chi_{\lambda + \mu_M}, M \in \Upsilon$ are all different, there exist $K' \in U^0, M_0 \in \Upsilon$ such

that $\chi_{\lambda+\mu_M}(K') = 1$ if $M = M_0$; 0 otherwise. Hence $f(F^\phi K' E^\psi) = \gamma_{M_0} v^{n_{M_0}} \in \mathcal{A}_0$, hence $\gamma_{M_0} \in \mathcal{A}_0$. \square

Let $\ell \geq n$ (the Coxeter number of \mathfrak{sl}_n) be an odd integer, and let q be a primitive ℓ -th root of 1. Let $p_\ell(v)$ denote the ℓ -th cyclotomic polynomial, then we have an isomorphism of fields $\mathcal{A}_0/(p_\ell(v)) \cong \mathbb{Q}(q)$. Set $U_q = U_{\mathcal{A}_0} \otimes_{\mathcal{A}_0} \mathbb{Q}(q)$ and $O_q = O \otimes_{\mathcal{A}_0} \mathbb{Q}(q)$. They are both Hopf algebras over $\mathbb{Q}(q)$ and inherit the comultiplications, counits and antipodes from $U_{\mathcal{A}_0}$ and O respectively. We denote the images of $E_i^{(N)}, F_i^{(N)}, K_i^{\pm 1}, \begin{bmatrix} K_i; c \\ t \end{bmatrix} \in U_{\mathcal{A}_0}$ in U_q by the same notations.

Proposition 2.2. *There is a pairing of Hopf algebras $(,): U_q \times O_q \rightarrow \mathbb{Q}(q)$, and it induces an embedding $O_q \hookrightarrow U_q^*$.*

Proof. This is basically Lemma 6.1 of [CL], where the specialization of the larger \mathcal{A}_0 -algebra $\Gamma(\mathfrak{g})$, i.e. $\Gamma_q = \Gamma(\mathfrak{g}) \otimes_{\mathcal{A}_0} \mathbb{Q}(q)$, is considered, and the pairing between Γ_q and O_q is non-degenerate. However if instead we consider the pairing between U_q and O_q , it is only non-degenerate on the O_q -half, i.e. $(u, f) = 0$ for all $u \in U_q$ implies that $f = 0 \in O_q$. To see why it is degenerate on the U_q -half, consider the image of $K_i^\ell - 1 \in U_{\mathcal{A}_0}$ in U_q , denoted by the same notation. It is not difficult to see that $(K_i^\ell - 1, f) = 0$ for all $f \in O_q$, but $K_i^\ell \neq 1$ in U_q (instead $K_i^{2\ell} = 1$ in U_q). The injectivity of the induced map $O_q \rightarrow U_q^*$ can also be proved using the arguments of Proposition 2.1. \square

The dual space U_q^* admits two commuting (left) actions of U_q , which we denote by ρ_1, ρ_2 . By definition, $\rho_1(u)f(u') = f(u'u)$ and $\rho_2(u)f(u') = f(S(u)u')$ for any $f \in U_q^*, u, u' \in U_q$, where S denotes the antipode of U_q . The Hopf algebra O_q is a $U_q \times U_q$ -submodule of U_q^* , and the U_q -actions can be expressed as follows:

$$\rho_1(u)g = \sum g_{(1)}(u, g_{(2)}), \quad \rho_2(u)g = \sum (S(u), g_{(1)})g_{(2)},$$

for any $u \in U_q, g \in O_q$, where $\Delta(g) = \sum g_{(1)} \otimes g_{(2)}$. The question we want to investigate is how O_q decomposes as a $U_q \times U_q$ -module, i.e. as a bicomodule of itself.

Let V be a finite dimensional representation of U_q , and let V^* be the dual representation defined by $uf(v) = f(S(u)v)$ for any $u \in U_q, f \in V^*, v \in V$. It is obvious that the map $\phi_V: V \otimes V^* \rightarrow U_q^*; v \otimes f \mapsto f(\cdot v)$ is a $U_q \times U_q$ -morphism. We denote the image of ϕ_V by $\mathbb{M}(V)$, called the matrix coefficients of V . Usually ϕ_V is not injective unless V is irreducible.

Lemma 2.3. *Let V, V' be finite-dimensional U_q -modules and $U \subset V$ be a submodule, then we have the following:*

- (1) $\phi_V(U \otimes V^*) = \mathbb{M}(U)$ and $\phi_V(V \otimes (V/U)^*) = \mathbb{M}(V/U)$. In particular $\mathbb{M}(U), \mathbb{M}(V/U) \subset \mathbb{M}(V)$.
- (2) $\mathbb{M}(V \otimes V') = \mathbb{M}(V) \cdot \mathbb{M}(V')$ and $\mathbb{M}(V \oplus V') = \mathbb{M}(V) + \mathbb{M}(V')$.
- (3) $\mathbb{M}(V^*) = S(\mathbb{M}(V))$.

Proof. (1) is easy to prove: in terms of matrix representations, $\mathbb{M}(U)$ and $\mathbb{M}(V/U)$ are the matrix coefficients in the diagonal blocks. Also note that the multiplication and the map S of U_q^* are defined by taking the transposes of the comultiplication and the antipode of U_q . Hence (2) and (3) follow. \square

We have a triangular decomposition $U_q = U_q^- U_q^0 U_q^+$, where $U_q^- = U^- \otimes_{\mathcal{A}} \mathbb{Q}(q)$ and similar definitions for U_q^0 and U_q^+ . Denote by X, X^+ the weight lattice and the dominant weights of \mathfrak{g} . For $\lambda \in X$, the character $\chi_\lambda : U^0 \rightarrow \mathcal{A}$ induces a character of U_q^0 to $\mathbb{Q}(q)$: $K_i^{\pm 1} \mapsto q^{\pm \langle \lambda, \alpha_i^\vee \rangle}$, $\begin{bmatrix} K_i; c \\ t \end{bmatrix} \mapsto \begin{bmatrix} \langle \lambda, \alpha_i^\vee \rangle + c \\ t \end{bmatrix}_q$, where the subscript q

means evaluating an element of $\mathbb{Z}[v, v^{-1}]$ at $v = q$. Let \mathcal{C}_f be the category of finite dimensional U_q -modules with a weight decomposition with respect to U_q^0 . We will show that $O_q \subset U_q^*$ is precisely the linear span of matrix coefficients of modules from \mathcal{C}_f . To prove it, let's first recall some important modules in \mathcal{C}_f . For any dominant weight $\lambda \in X^+$, we can associate four canonical modules: the Weyl module V_λ , the dual of the Weyl module V_λ^* , the irreducible module L_λ and the tilting module T_λ .

The Weyl module V_λ is generated by a vector of highest weight λ , and has the universal property that any module in \mathcal{C}_f generated by a vector of highest weight λ is a quotient of V_λ . The character of V_λ is given by Weyl's character formula (see [APW, A2] for definitions of V_λ and V_λ^* in terms of some induction functor and its derived functors). We have the following property for these standard objects: $\text{Ext}_{\mathcal{C}_f}^i(V_\lambda, V_\mu^*) = \mathbb{Q}(q)$ if $i = 0$ and $\lambda = -\omega_0 \mu$; 0 otherwise, where ω_0 is the longest element in the Weyl group.

The irreducible module L_λ is the head of V_λ as well as the socle of $V_{-\omega_0 \lambda}^*$, and $L_\lambda^* \cong L_{-\omega_0 \lambda}$. Furthermore the modules $L_\lambda, \lambda \in X^+$ give a complete list of non-isomorphic irreducible modules in \mathcal{C}_f .

A module in \mathcal{C}_f is called tilting if it admits both a Weyl filtration and a dual Weyl filtration. Tilting modules are closed by taking the dual and the tensor product. For each $\lambda \in X^+$, there exists a unique (up to isomorphism) indecomposable tilting module T_λ such that T_λ admits a Weyl filtration starting with $V_\lambda \hookrightarrow T_\lambda$, and any other Weyl modules V_μ entering the Weyl filtration of T_λ satisfy that $\mu < \lambda$, here \leq is the usual partial order on X determined by a set of positive roots. The highest weight λ occurs with multiplicity 1 in T_λ . By consideration of characters, we have $T_\lambda^* \cong T_{-\omega_0 \lambda}$. Moreover the modules $T_\lambda, \lambda \in X^+$ form a complete list of inequivalent indecomposable tilting modules. It is easy to see that $T_\lambda = V_\lambda$ if and only if V_λ is irreducible. There are enough projectives in \mathcal{C}_f and all projective modules are tilting (see [APW], [A2]).

Proposition 2.4. $O_q \subset U_q^*$ is the linear span of matrix coefficients of modules from \mathcal{C}_f , i.e. $O_q = \sum_{V \in \mathcal{C}_f} \mathbb{M}(V)$.

Proof. By Proposition 2.1, the $\mathbb{Q}(q)$ -algebra O_q is generated by X_{ij} , the matrix coefficients of V_{ω_1} . By Lemma 2.3 (2), we have $O_q = \sum_n \mathbb{M}(V_{\omega_1}^{\otimes n}) \subset \sum_{V \in \mathcal{C}_f} \mathbb{M}(V)$. To prove the inverse inclusion, it suffices to show that O_q contains the matrix coefficients of all the tilting modules, since all projective modules are tilting.

Note that the Weyl modules $V_{\omega_i}, i = 1, \dots, n-1$ associated to the fundamental weights are all irreducible, since the weights occurring in V_{ω_i} lie in the W -orbit of ω_i with multiplicity 1, hence we must have $L_{\omega_i} = V_{\omega_i} = T_{\omega_i}$. It is well known that the fundamental representation of $\mathfrak{sl}(n, \mathbb{C})$ with highest weight ω_i can be realized as the i -th exterior power of the n -dimensional natural representation, in particular it is a direct summand of the i -th tensor power. Therefore by consideration of the characters, V_{ω_i} is either a direct summand of $V_{\omega_1}^{\otimes i}$ or a composition factor of a direct summand of

$V_{\omega_1}^{\otimes i}$ (which is tilting). Either way, we have $\mathbb{M}(T_{\omega_i}) \subset \mathbb{M}(V_{\omega_1})^i$ by Lemma 2.3. Now for any $\lambda \in X^+$ with $m_i = \langle \lambda, \alpha_i^\vee \rangle$, the tilting module T_λ must be a direct summand of $T_{\omega_1}^{\otimes m_1} \otimes \cdots \otimes T_{\omega_{n-1}}^{\otimes m_{n-1}}$, hence $\mathbb{M}(T_\lambda) \subset \mathbb{M}(T_{\omega_1})^{m_1} \cdots \mathbb{M}(T_{\omega_{n-1}})^{m_{n-1}} \subset \sum_n \mathbb{M}(V_{\omega_1})^n$. \square

3. AN INCREASING FILTRATION OF O_q

Fix $\mathfrak{g} = \mathfrak{sl}_n$, and q to be a primitive ℓ -th root of 1, where $\ell \geq n$ is odd. It follows from Proposition 2.1 that the quantum coordinate algebra O_q is generated by X_{ij} , $1 \leq i, j \leq n$ over $\mathbb{Q}(q)$ subject to a list of relations. Proposition 2.4 identifies O_q with the linear span of matrix coefficients of finite dimensional U_q -modules. In this section, we will describe a canonical increasing filtration of O_q as a $U_q \times U_q$ -module.

Let $R, R^+, X, X^+, \mathcal{W}$ denote the root system, positive roots, weight lattice, dominant weights and Weyl group of \mathfrak{g} . The affine Weyl group \mathcal{W}_ℓ is generated by the affine reflections $s_{\beta, m}, \beta \in R^+, m \in \mathbb{Z}$ given by

$$s_{\beta, m} \cdot \lambda = s_\beta \cdot \lambda + m\ell\beta, \quad \lambda \in X.$$

Here s_β is the reflection corresponding to the positive root β , and we are using the dot-action defined by $s_\beta \cdot \lambda = s_\beta(\lambda + \rho) - \rho$, where ρ is the half sum of the positive roots.

Denote by C the first dominant alcove, i.e.

$$C = \{\lambda \in X^+ \mid \langle \lambda + \rho, \beta^\vee \rangle < \ell \text{ for all } \beta \in R^+\},$$

and set

$$\bar{C} = \{\lambda \in X \mid 0 \leq \langle \lambda + \rho, \beta^\vee \rangle \leq \ell \text{ for all } \beta \in R^+\},$$

then \bar{C} is a fundamental domain for the action of \mathcal{W}_ℓ on X .

The linkage principal (see [A1]) allows us to decompose any module from \mathcal{C}_f into summands corresponding to the representatives in \bar{C} , therefore it yields a decomposition of O_q as well.

Proposition 3.1. *As a $U_q \times U_q$ -module, we have $O_q \cong \bigoplus_{\lambda \in (\ell-1)\rho + \ell X^+} V_\lambda \otimes V_\lambda^* \oplus (\bigoplus_{\mu \in \bar{C} \setminus \{(\ell-1)\rho + \ell X\}} \Lambda_\mu)$, where $\Lambda_\mu = \sum_{\nu \in \mathcal{W}_\ell \cdot \mu \cap X^+} \mathbb{M}(T_\nu)$.*

Proof. Recall that O_q is spanned by the matrix coefficients of (tilting) modules from \mathcal{C}_f . The linkage principal implies that $O_q = \bigoplus_{\mu \in \bar{C}} \Lambda_\mu$, where $\Lambda_\mu = \sum_{\nu \in \mathcal{W}_\ell \cdot \mu \cap X^+} \mathbb{M}(T_\nu)$. The vertices of the simplex \bar{C} are $-\rho, \ell\omega_i - \rho, i = 1, \dots, n-1$, where ω_i 's are the fundamental weights of \mathfrak{g} . The \mathcal{W}_ℓ -orbits of these vertices consist of weights of the form $(\ell-1)\rho + \ell X$. By [APW, Corollary 7.6], if $\lambda \in (\ell-1)\rho + \ell X^+$, the Weyl module V_λ is irreducible, in which case $V_\lambda = T_\lambda$ and $\mathbb{M}(T_\lambda) \cong V_\lambda \otimes V_\lambda^*$. \square

Lemma 3.2. *Let $V, V' \in \mathcal{C}_f$.*

- (1) *Suppose V admits a Weyl filtration $0 = V^0 \subset V^1 \subset \cdots \subset V^m = V$ such that $V^i/V^{i-1} \cong V_{\lambda_i}$ for some $\lambda_i \in X^+$, then $\mathbb{M}(V) \subset \sum_i \mathbb{M}(T_{\lambda_i})$.*
- (2) *Suppose V' admits a dual Weyl filtration with factors isomorphic to $V_{\mu_i}^*$ for some $\mu_i \in X^+$, then $\mathbb{M}(V') \subset \sum_i \mathbb{M}(T_{\mu_i}^*)$.*

Proof. Let f_i be the composition of $V^i \twoheadrightarrow V^i/V^{i-1} \cong V_{\lambda_i} \hookrightarrow T_{\lambda_i}$. Apply $\text{Hom}_{\mathcal{C}_f}(-, T_{\lambda_i})$ to the short exact sequence $0 \rightarrow V^i \rightarrow V \rightarrow V/V^i \rightarrow 0$, we get $\text{Hom}_{\mathcal{C}_f}(V, T_{\lambda_i}) \rightarrow \text{Hom}_{\mathcal{C}_f}(V^i, T_{\lambda_i}) \rightarrow \text{Ext}_{\mathcal{C}_f}(V/V^i, T_{\lambda_i})$ which is exact. Since V/V^i and T_{λ_i} admit a Weyl filtration and a dual Weyl filtration respectively, it follows that $\text{Ext}_{\mathcal{C}_f}(V/V^i, T_{\lambda_i}) = 0$,

hence $\text{Hom}_{\mathcal{C}_f}(V, T_{\lambda_i}) \rightarrow \text{Hom}_{\mathcal{C}_f}(V^i, T_{\lambda_i})$ is surjective. Let $g_i : V \rightarrow T_{\lambda_i}$ be a preimage of f_i , then $V^i \cap \text{Ker } g_i = V^{i-1}$. Define $g = \sum g_i : V \rightarrow \oplus_i T_{\lambda_i}$, then $\text{Ker } g = \cap_i \text{Ker } g_i = 0$, i.e. g is injective. Hence by Lemma 2.3 we have $\mathbb{M}(V) \subset \sum_i \mathbb{M}(T_{\lambda_i})$. Analogously we can prove (2) by constructing a surjective map $\oplus_i T_{\mu_i}^* \rightarrow V'$, but we can also argue as follows: by assumption, V'^* admits a Weyl filtration with factors isomorphic to V_{μ_i} , hence by (1) we have $\mathbb{M}(V'^*) \subset \sum \mathbb{M}(T_{\mu_i})$, hence it follows from Lemma 2.3 (3) that $\mathbb{M}(V') = S(\mathbb{M}(V'^*)) \subset \sum_i S(\mathbb{M}(T_{\mu_i})) = \sum_i \mathbb{M}(T_{\mu_i}^*)$. \square

Theorem 3.3. *Let $\mu \in \bar{C} \setminus \{(\ell-1)\rho + \ell X\}$, and write $\mathcal{W}_\ell \cdot \mu \cap X^+ = \{\nu_i, i \geq 1\}$ so that $\nu_i \leq \nu_j$ implies $i \leq j$. Set $P^i = \sum_{j \leq i} \mathbb{M}(T_{\nu_j})$, then $P^1 \subset \dots \subset P^{i-1} \subset P^i \subset \dots$ is an increasing filtration of $U_q \times U_q$ -submodules of Λ_μ with subquotients $P^i/P^{i-1} \cong V_{-\omega_0\nu_i}^* \otimes V_{\nu_i}^*$ as a $U_q \times U_q$ -module.*

Proof. Since the dual Weyl filtration of T_{ν_i} ends with $T_{\nu_i} \twoheadrightarrow V_{-\omega_0\nu_i}^*$, there exists a submodule $W \subset T_{\nu_i}$ such that $T_{\nu_i}/W \cong V_{-\omega_0\nu_i}^*$, and W admits a filtration with factors isomorphic to V_γ^* 's with $-\omega_0\gamma < \nu_i$ and $-\omega_0\gamma \in \mathcal{W}_\ell \cdot \nu_i$, i.e. $-\omega_0\gamma = \nu_j$ for some $j < i$. Hence we have $\phi_{T_{\nu_i}}(W \otimes T_{\nu_i}^*) = \mathbb{M}(W) \subset P^{i-1}$ by Lemma 2.3 and Lemma 3.2; analogously we also have $\phi_{T_{\nu_i}}(T_{\nu_i} \otimes (T_{\nu_i}/V_{\nu_i})^*) = \mathbb{M}(T_{\nu_i}/V_{\nu_i}) \subset P^{i-1}$. Set $N = W \otimes T_{\nu_i}^* + T_{\nu_i} \otimes (T_{\nu_i}/V_{\nu_i})^*$, then $\phi_{T_{\nu_i}}$ induces a surjective map $\psi : (T_{\nu_i} \otimes T_{\nu_i}^*)/N \rightarrow \mathbb{M}(T_{\nu_i})/(\mathbb{M}(T_{\nu_i}) \cap P^{i-1}) = P^i/P^{i-1}$. Note that $(T_{\nu_i} \otimes T_{\nu_i}^*)/N \cong V_{-\omega_0\nu_i}^* \otimes V_{\nu_i}^*$, the socle of which is $L_{\nu_i} \otimes L_{\nu_i}^*$. Since $\psi(L_{\nu_i} \otimes L_{\nu_i}^*) = (\mathbb{M}(L_{\nu_i}) + P^{i-1})/P^{i-1} \cong \mathbb{M}(L_{\nu_i}) \neq 0$, ψ is also injective, hence it induces the isomorphism $V_{-\omega_0\nu_i}^* \otimes V_{\nu_i}^* \xrightarrow{\sim} P^i/P^{i-1}$. \square

As an application, we will compute $HH^0(O_q, O_q)$, the 0-th Hochschild cohomology of the coalgebra O_q with coefficients in O_q , which is equivalent to the algebra of cocommutative elements in O_q .

Suppose $f \in O_q$ is cocommutative, then $f(uu') = f(u'u)$ for any $u, u' \in U_q$, i.e. $\rho_1(u)f = \rho_2(S^{-1}u)f$.

Lemma 3.4. *For any $\lambda \in X^+$, the subspace $Y = \{y \in V_{-\omega_0\lambda}^* \otimes V_\lambda^* : \rho'_1(u)y = \rho'_2(S^{-1}u)y, \forall u \in U_q\}$ is one-dimensional, where ρ'_1, ρ'_2 denote the actions of U_q on $V_{-\omega_0\lambda}^*$ and V_λ^* respectively.*

Proof. Set $B_q = U_q^0 U_q^-$, $k = \mathbb{Q}(q)$, and denote by k_λ the one-dimensional B_q -module defined by the character $\chi_\lambda : U_q^0 \rightarrow k$ and extended to a B_q -module with trivial U_q^- -action. Recall from [APW, A1] that $V_{-\omega_0\lambda}^*$ is an integrable submodule of $\text{Hom}_{B_q}(U_q, k_\lambda)$, where U_q is considered a B_q -module via left multiplication of B_q on U_q , and the U_q -module structure on $\text{Hom}_{B_q}(U_q, k_\lambda)$ is defined via the right multiplication of U_q on itself. Choose a basis v_1, \dots, v_s of $V_{-\omega_0\lambda}^*$ such that $v_1(uE_i^{(r)}) = 0$ for any i if $r > 0$ and $v_1(b) = \chi_\lambda(b)$ for any $b \in B_q$. Then v_1 has weight λ (recall that λ occurs with multiplicity 1 in $V_{-\omega_0\lambda}^*$). Assume that v_2, \dots, v_s are also homogeneous vectors (with weights less than λ), then $v_i(b) = 0$ for any $b \in U_q^-$ and $i = 2, \dots, s$. Similarly choose a homogeneous basis v'_1, \dots, v'_s of V_λ^* such that v'_1 has weight $-\lambda$. For any $y = \sum_{ij} y_{ij} v_i \otimes v'_j \in Y$, it is easy to check that in order for y to satisfy the equality $\rho'_1(u)y = \rho'_2(S^{-1}u)y$ for all $u = u^0 \in U_q^0$, we must have $y_{1i} = 0$ for any $i \neq 1$ (since $\chi_{-\lambda} S^{-1} = \chi_\lambda$). Define a linear map $pr : Y \rightarrow k$; $y \mapsto y_{11}$, which we will show is in fact injective. Suppose $y_{11} = 0$ for some $y \in Y$, then for any $u_1^+, u_2^+ \in$

U_q^+ , we have $y(u_1^+ \otimes u_2^+) = \sum_{ij} y_{ij} v_i(u_1^+) v_j'(u_2^+) = \sum_{ij} y_{ij} \{\rho_1'(u_1^+) v_i\} (1) v_j'(u_2^+) = \sum_{ij} y_{ij} v_i(1) \{\rho_2'(S^{-1}u_1^+) v_j'\} (u_2^+) = y_{11} v_1'(u_2^+ S^{-1}u_1^+) = 0$, hence $y = 0$. It means that pr is injective, hence $\dim_k Y \leq 1$. On the other hand, let e_i be a basis of L_λ and let δ_i be the dual basis of L_λ^* , it is easy to check that $\sum_i e_i \otimes \delta_i \in L_\lambda \otimes L_\lambda^* \subset V_{-\omega_0\lambda}^* \otimes V_\lambda^*$ satisfies the condition of Y , hence $\dim_k Y = 1$. \square

Proposition 3.5. $HH^0(O_q, O_q) \cong \mathbb{Q}(q)[X]^\mathcal{W}$.

Proof. For a module $V \in \mathcal{C}_f$, we denote by $[V]$ its image in the Grothendieck ring $[\mathcal{C}_f]$. It is clear that $[\mathcal{C}_f]$ is isomorphic to $\mathbb{Z}[X]^\mathcal{W}$, with the isomorphism given by $[V] \rightarrow \text{ch} V$. Let $\mathcal{R} = [\mathcal{C}_f] \otimes_{\mathbb{Z}} \mathbb{Q}(q)$, then $\mathcal{R} \cong \mathbb{Q}(q)[X]^\mathcal{W}$ and it has a natural basis of simple characters $\{\text{ch} L_\lambda, \lambda \in X^+\}$.

For $V \in \mathcal{C}_f$, define the trace of V as $\text{tr}_V = \phi_V(\sum_i v_i \otimes f_i) \in \mathbb{M}(V)$, where $\{v_i\}$ is a basis of V and $\{f_i\}$ is the dual basis of V^* . Let $\mathbf{tr} \subset O_q$ be the $\mathbb{Q}(q)$ -linear span of traces of modules from \mathcal{C}_f . If $U \hookrightarrow V \rightarrow W$ is a short exact sequence of modules from \mathcal{C}_f , we have $\text{tr}_V = \text{tr}_U + \text{tr}_W$, therefore each tr_V can be written as a linear combination of traces of its composition factors. Note that $\text{tr}_\lambda := \text{tr}_{L_\lambda}$, $\lambda \in X^+$, are linearly independent, hence they form a basis of \mathbf{tr} , and $\mathbf{tr} \cong \mathcal{R}$ as a vector space. Since $\text{tr}_{V \otimes V'} = \text{tr}_V \text{tr}_{V'}$, it is in fact an isomorphism of algebras.

Denote by \mathbf{Co} the set of elements in O_q that are cocommutative, it suffices to show that $\mathbf{Co} = \mathbf{tr}$. It is obvious that $\mathbf{tr} \subset \mathbf{Co}$. To prove the inverse inclusion, define $P^\lambda = \sum_{\mu \leq \lambda, \mu \in X^+} \mathbb{M}(T_\mu) \subset O_q$. Since $O_q = \bigcup_{\lambda \in X^+} P^\lambda$, it suffices to prove that $P^\lambda \cap \mathbf{Co} \subset \mathbf{tr}$. If λ is minimal (for the ordering \leq) among the weights in X^+ , then $P^\lambda = \mathbb{M}(T_\lambda) = \mathbb{M}(L_\lambda) \cong L_\lambda \otimes L_\lambda^*$. It follows from Lemma 3.4 that $P^\lambda \cap \mathbf{Co} = \mathbb{Q}(q) \text{tr}_\lambda \subset \mathbf{tr}$. Now assume that $P^\mu \cap \mathbf{Co} \subset \mathbf{tr}$ is true for any $\mu < \lambda, \mu \in X^+$. From the proof of Theorem 3.3, we have $P^\lambda / \sum_{\mu < \lambda, \mu \in X^+} \mathbb{M}(T_\mu) \cong V_{-\omega_0\lambda}^* \otimes V_\lambda^*$. Suppose $f \in P^\lambda \cap \mathbf{Co}$, then $\rho_1(u)f = \rho_2(S^{-1}u)f$ for any $u \in U_q$, hence the image of f in $P^\lambda / \sum_{\mu < \lambda, \mu \in X^+} \mathbb{M}(T_\mu)$ belongs to the subspace Y defined in Lemma 3.4. Since Y is one-dimensional and is spanned by the image of the trace of L_λ , there exists a scalar ζ such that $f - \zeta \text{tr}_\lambda \in \sum_{\mu < \lambda, \mu \in X^+} \mathbb{M}(T_\mu) \cap \mathbf{Co}$. By induction $f - \zeta \text{tr}_\lambda \in \mathbf{tr}$, hence $f \in \mathbf{tr}$. \square

It is well known that the category of finite dimensional representations of U_q is semisimple when q is not a root of unity, in which case the quantum function algebra O_q is the direct sum of matrix coefficients of irreducible modules, and all the cocommutative elements of O_q come from the traces of finite dimensional modules. Proposition 3.5 says that the last statement is also true at roots of 1.

Remark 3.6. For other types of simple Lie algebras, I am not sure if O_q is linearly spanned by the matrix coefficients of finite dimensional U_q -modules. Nonetheless if we denote the latter by O'_q , then obviously $O_q \subset O'_q$, and the results in this section hold for O'_q .

4. THE CASE OF \mathfrak{sl}_2

In this section we study the \mathfrak{sl}_2 case more thoroughly. Let $\ell > 2$ be odd, q be a primitive ℓ -th root of unity. The quantum function algebra O_q is generated by a, b, c, d over $\mathbb{Q}(q)$ subject to the relations:

$$ab = qba, \quad ac = qca,$$

$$\begin{aligned} bd &= qdb, & cd &= qdc, \\ bc &= cb, & ad - qbc &= da - q^{-1}bc = 1. \end{aligned}$$

The comultiplication Δ , counit ε and antipode S are defined by

$$\begin{aligned} \Delta(a) &= a \otimes a + b \otimes c, & \Delta(b) &= a \otimes b + b \otimes d, \\ \Delta(c) &= c \otimes a + d \otimes c, & \Delta(d) &= c \otimes b + d \otimes d, \\ \varepsilon(a) &= \varepsilon(d) = 1, & \varepsilon(b) &= \varepsilon(c) = 0, \\ S(a) &= d, & S(d) &= a, & S(b) &= -q^{-1}b, & S(c) &= -qc. \end{aligned}$$

The quantum group U_q is generated by $E^{(i)}, F^{(i)}, K^{\pm 1}, \begin{bmatrix} K; c \\ t \end{bmatrix}$ subject to some relations.

For $\mathfrak{g} = \mathfrak{sl}_2$, we have $X = \mathbb{Z}, X^+ = \mathbb{N}$. The Weyl module V_n , for $n \in \mathbb{N}$, is $(n+1)$ -dimensional, with a basis f_0, f_1, \dots, f_n such that f_i is of weight $-n+2i$ and

$$E^{(j)} f_i = \begin{bmatrix} i+j \\ i \end{bmatrix}_q f_{i+j}, \quad F^{(j)} f_i = \begin{bmatrix} n-i+j \\ j \end{bmatrix}_q f_{i-j}.$$

The dual representation V_n^* is also $(n+1)$ -dimensional, with a basis e_0, e_1, \dots, e_n such that e_i is of weight $n-2i$ and

$$E^{(j)} e_i = \begin{bmatrix} i \\ j \end{bmatrix}_q e_{i-j}, \quad F^{(j)} e_i = \begin{bmatrix} n-i \\ j \end{bmatrix}_q e_{i+j}.$$

The Weyl modules V_n and their duals V_n^* are reducible in general, but their composition series are well-known, so are the Weyl filtrations of the tilting modules T_n .

Lemma 4.1. *Write $n = n_0 + \ell n_1$ with $0 \leq n_0 \leq \ell - 1, n_1 \geq 0$, then*

- (1) *if $n_1 = 0$ or $n_0 = \ell - 1$, V_n is irreducible, hence $T_n = V_n = V_n^* = L_n$;*
- (2) *assume now that $0 \leq n_0 \leq \ell - 2$ and $n_1 \geq 1$, set $n' = (\ell - 2 - n_0) + \ell(n_1 - 1)$, then we have the following exact sequences: $L_{n'} \hookrightarrow V_n \twoheadrightarrow L_n$, $L_n \hookrightarrow V_n^* \twoheadrightarrow L_{n'}$, $V_n \hookrightarrow T_n \twoheadrightarrow V_{n'}$ and $V_{n'}^* \hookrightarrow T_n \twoheadrightarrow V_n^*$.*

Proof. See [L2, Proposition 9.2] or [APW, Corollary 4.6] for assertions about V_n, V_n^* . Since $-\omega_0 n = -(-1)n = n$, we have $L_n^* \cong L_n$ and $T_n^* = T_n$. By [A2, Proposition 5.8], T_n is the projective cover of $L_{n'}$. Since $\text{Ext}_{\mathcal{C}_f}^i(V_m, V_k^*) = \mathbb{Q}(q)$ if $i = 0$ and $m = k$; 0 otherwise, we have the following reciprocity of multiplicities: $(T_n, V_k) = \dim_{\mathbb{Q}(q)} \text{Hom}_{\mathcal{C}_f}(T_n, V_k^*) = (V_k^*, L_{n'})$. Hence the composition factors of the Weyl modules imply the Weyl filtrations of the tilting modules. \square

Let $\mathcal{W} \cong \mathbb{Z}_2 = \{1, -1\}$ be the Weyl group of \mathfrak{sl}_2 , and let $\mathcal{W}_\ell \cong \mathbb{Z}_2 \ltimes \mathbb{Z}$ be the affine Weyl group. The shifted action of \mathcal{W}_ℓ on $X = \mathbb{Z}$ is defined by: $(1, m) \cdot n = n + 2m\ell$; $(-1, m) \cdot n = -n - 2 + 2m\ell$. The fundamental domain for \mathcal{W}_ℓ is given by $\bar{C} = \{-1, 0, \dots, \ell - 1\}$, and the linkage principal yields the following decomposition of O_q .

Proposition 4.2. *$O_q = (\oplus_{k \geq 1} V_{k\ell-1} \otimes V_{k\ell-1}) \oplus (\oplus_{m=0}^{\ell-2} \Lambda_m)$ as a $U_q \times U_q$ -module, where $\Lambda_m = \sum_{s \in \mathcal{W}_\ell \cdot m, s \geq 0} \mathbb{M}(T_s)$.*

Proof. See Proposition 3.1 and Lemma 4.1. \square

To analyze the structure of Λ_m , $0 \leq m \leq \ell - 2$ even further, let us take a closer look at the bimodule structure of each $\mathbb{M}(T_s)$.

We say that $n_1 < \cdots < n_i < n_{i+1} < \cdots$ is a sequence if $n_i = n'_{i+1}$ for any $i \geq 1$ ($n_i \geq 0, n_i \not\equiv -1 \pmod{\ell}$ is assumed). We can form $\ell - 1$ sequences of infinite length starting with $0, 1, \dots, \ell - 2$ respectively, which is the same as to arrange the weights in the \mathcal{W}_ℓ -orbits of $0, 1, \dots, \ell - 2$ in an increasing order.

A module of finite length is called rigid if the socle and radical series coincide, in which case the unique shortest filtration with semisimple quotients is called the Loewy series. We represent the structure of rigid modules pictorially, with the top blocks corresponding to the tops of the modules and the bottom blocks representing the socles.

Lemma 4.3. *Let $n_1 < n_2$ be a sequence, i.e. $n_1 = n'_2$, then*

- (1) $\mathbb{M}(L_{n_1}) = L_{n_1} \otimes L_{n_1}$ and $\mathbb{M}(L_{n_2}) = L_{n_2} \otimes L_{n_2}$.
- (2) $\mathbb{M}(V_{n_2})$ (resp. $\mathbb{M}(V_{n_2}^*)$) is rigid and the Loewy series is given by $0 \subset \mathbb{M}(L_{n_1}) \oplus \mathbb{M}(L_{n_2}) \subset \mathbb{M}(V_{n_2})$ (resp. $0 \subset \mathbb{M}(L_{n_1}) \oplus \mathbb{M}(L_{n_2}) \subset \mathbb{M}(V_{n_2}^*)$) with layers depicted by

$$\begin{aligned} \mathbb{M}(V_{n_2}) &\sim \begin{array}{c} \boxed{L_{n_2} \otimes L_{n_1}} \\ \boxed{L_{n_1} \otimes L_{n_1}} \oplus \boxed{L_{n_2} \otimes L_{n_2}} \end{array} \\ (\text{resp.} \\ \mathbb{M}(V_{n_2}^*) &\sim \begin{array}{c} \boxed{L_{n_1} \otimes L_{n_2}} \\ \boxed{L_{n_1} \otimes L_{n_1}} \oplus \boxed{L_{n_2} \otimes L_{n_2}} \end{array}). \end{aligned}$$

Proof. (1) is obvious. Tensoring $0 \subset L_{n_1} \subset V_{n_2}$ together with $0 \subset L_{n_2} = \text{Ann}(L_{n_1}) \subset V_{n_2}^*$, we obtain a filtration of $V_{n_2} \otimes V_{n_2}^*$: $0 \subset L_{n_1} \otimes \text{Ann}(L_{n_1}) \subset L_{n_1} \otimes V_{n_2}^* + V_{n_2} \otimes \text{Ann}(L_{n_1}) \subset V_{n_2} \otimes V_{n_2}^*$. Recall the $U_q \times U_q$ -map $\phi_{V_{n_2}} : V_{n_2} \otimes V_{n_2}^* \rightarrow \mathbb{M}(V_{n_2})$, it is easy to see that $\text{Ker } \phi_{V_{n_2}} = L_{n_1} \otimes \text{Ann}(L_{n_1})$; $\phi_{V_{n_2}}(L_{n_1} \otimes V_{n_2}^*) = \mathbb{M}(L_{n_1})$; and $\phi_{V_{n_2}}(V_{n_2} \otimes \text{Ann}(L_{n_1})) = \mathbb{M}(L_{n_2})$. It follows that $\mathbb{M}(V_{n_2})$ admits the filtration as claimed in (2). The constituent $L_{n_2} \otimes L_{n_1}$ is nontrivially linked with both $L_{n_1} \otimes L_{n_1}$ and $L_{n_2} \otimes L_{n_2}$, since the exact sequence $L_{n_1} \hookrightarrow V_{n_2} \twoheadrightarrow L_{n_2}$ does not split. Similar arguments apply to $\mathbb{M}(V_{n_2}^*)$. \square

Lemma 4.4. *Let $n_1 < n_2$ be a sequence and $0 \leq n_1 \leq \ell - 2$, then*

- (1) $\mathbb{M}(T_{n_2})$ is rigid and indecomposable as a $U_q \times U_q$ -module. The Loewy series is given by $0 \subset \mathbb{M}(L_{n_1}) \oplus \mathbb{M}(L_{n_2}) \subset \mathbb{M}(V_{n_2}) + \mathbb{M}(V_{n_2}^*) \subset \mathbb{M}(T_{n_2})$ with layers depicted by

$$\mathbb{M}(T_{n_2}) \sim \begin{array}{c} \boxed{L_{n_1} \otimes L_{n_1}} \\ \boxed{L_{n_2} \otimes L_{n_1}} \oplus \boxed{L_{n_1} \otimes L_{n_2}} \\ \boxed{L_{n_1} \otimes L_{n_1}} \oplus \boxed{L_{n_2} \otimes L_{n_2}} \end{array}.$$

- (2) $\mathbb{M}(T_{n_1}) \subset \mathbb{M}(T_{n_2})$ and $\mathbb{M}(T_{n_2})/\mathbb{M}(T_{n_1}) \cong V_{n_2}^* \otimes V_{n_2}^*$.

Proof. Tensoring $0 \subset L_{n_1} \subset V_{n_2} \subset T_{n_2}$ together with $0 \subset \text{Ann}(V_{n_2}) \subset \text{Ann}(L_{n_1}) \subset T_{n_2}^*$ gives a filtration of $T_{n_2} \otimes T_{n_2}^*$; applying $\phi_{T_{n_2}} : T_{n_2} \otimes T_{n_2}^* \rightarrow \mathbb{M}(T_{n_2})$ to it, we

obtain the desired filtration for $\mathbb{M}(T_{n_2})$. Choose a basis of T_{n_2} so that the matrix representations with respect to this basis look like

$$\begin{pmatrix} \mathbb{M}(L_{n_1}) & \Delta & \star \\ 0 & \mathbb{M}(L_{n_2}) & \nabla \\ 0 & 0 & \mathbb{M}(L_{n_1}) \end{pmatrix}.$$

The diagonal blocks correspond to the simple layers of T_{n_2} ; the matrix coefficients $\mathbb{M}(L_{n_1})$ and $\mathbb{M}(L_{n_2})$, together with Δ (resp. ∇), span $\mathbb{M}(V_{n_2})$ (resp. $\mathbb{M}(V_{n_2}^*)$); the coefficients in \star generate the whole $\mathbb{M}(T_{n_2})$. It is not hard to see that $\mathbb{M}(T_{n_2})$ is indeed rigid and indecomposable.

It is obvious that $\mathbb{M}(T_{n_1}) \subset \mathbb{M}(T_{n_2})$ since $T_{n_1} \cong L_{n_1}$ for $0 \leq n_1 \leq \ell - 2$. Moreover $\mathbb{M}(T_{n_2})/\mathbb{M}(T_{n_1}) \cong (T_{n_2} \otimes T_{n_2}^*)/\phi_{T_{n_2}}^{-1} \mathbb{M}(L_{n_1}) = (T_{n_2} \otimes T_{n_2}^*)/(L_{n_1} \otimes T_{n_2}^* + T_{n_2} \otimes \text{Ann}(V_{n_2})) \cong V_{n_2}^* \otimes V_{n_2}^*$. \square

Lemma 4.5. *Let $n_1 < n_2 < n_3$ be a sequence, then*

- (1) $\mathbb{M}(T_{n_3})$ is rigid and indecomposable. The Loewy series is given by $0 \subset \mathbb{M}(L_{n_1}) \oplus \mathbb{M}(L_{n_2}) \oplus \mathbb{M}(L_{n_3}) \subset \mathbb{M}(V_{n_2}) + \mathbb{M}(V_{n_2}^*) + \mathbb{M}(V_{n_3}) + \mathbb{M}(V_{n_3}^*) \subset \mathbb{M}(T_{n_3})$ with layers depicted by

$$M(T_{n_3}) \sim \begin{array}{c} \boxed{L_{n_2} \otimes L_{n_1}} \oplus \boxed{L_{n_2} \otimes L_{n_2}} \oplus \boxed{L_{n_3} \otimes L_{n_2}} \oplus \boxed{L_{n_2} \otimes L_{n_3}} \\ \boxed{L_{n_1} \otimes L_{n_1}} \oplus \boxed{L_{n_2} \otimes L_{n_2}} \oplus \boxed{L_{n_3} \otimes L_{n_3}} \end{array}.$$

- (2) $\mathbb{M}(T_{n_2}) \cap \mathbb{M}(T_{n_3}) = \mathbb{M}(V_{n_2}) + \mathbb{M}(V_{n_2}^*)$. Moreover we have $\mathbb{M}(T_{n_3})/(\mathbb{M}(T_{n_2}) \cap \mathbb{M}(T_{n_3})) \cong V_{n_3}^* \otimes V_{n_3}^*$ and $\mathbb{M}(T_{n_3})/(\mathbb{M}(V_{n_3}) + \mathbb{M}(V_{n_3}^*)) \cong V_{n_2} \otimes V_{n_2}$.

Proof. The proof is parallel to the proof of the previous two lemmas. Since T_{n_3} is rigid with layers L_{n_2} , $L_{n_1} \oplus L_{n_3}$ and L_{n_2} from the socle to the top, we can choose a basis of T_{n_3} so that the matrix representations with respect to this basis look like

$$\begin{pmatrix} \mathbb{M}(L_{n_2}) & \Delta_{n_3} & \nabla_{n_2} & \star \\ 0 & \mathbb{M}(L_{n_3}) & 0 & \nabla_{n_3} \\ 0 & 0 & \mathbb{M}(L_{n_1}) & \Delta_{n_2} \\ 0 & 0 & 0 & \mathbb{M}(L_{n_2}) \end{pmatrix}.$$

We have the matrix coefficients of the irreducibles on the diagonal; Δ_{n_3} (resp. ∇_{n_3}) together with $\mathbb{M}(L_{n_2})$, $\mathbb{M}(L_{n_3})$ span $\mathbb{M}(V_{n_3})$ (resp. $\mathbb{M}(V_{n_3}^*)$); Δ_{n_2} (resp. ∇_{n_2}) together with $\mathbb{M}(L_{n_1})$, $\mathbb{M}(L_{n_2})$ span $\mathbb{M}(V_{n_2})$ (resp. $\mathbb{M}(V_{n_2}^*)$); the top \star generates the whole $\mathbb{M}(T_{n_3})$. Again it is not difficult to see that $\mathbb{M}(T_{n_3})$ is rigid and indecomposable, and the nonzero blocks in the matrix correspond to the layers of the Loewy series.

It's clear that $\mathbb{M}(T_{n_2}) \cap \mathbb{M}(T_{n_3}) = \mathbb{M}(V_{n_2}) + \mathbb{M}(V_{n_2}^*)$. Since $\phi_{T_{n_3}}^{-1}(\mathbb{M}(V_{n_2}) + \mathbb{M}(V_{n_2}^*)) = V_{n_2}^* \otimes T_{n_3}^* + T_{n_3} \otimes \text{Ann}(V_{n_3})$ and $\phi_{T_{n_3}}^{-1}(\mathbb{M}(V_{n_3}) + \mathbb{M}(V_{n_3}^*)) = V_{n_3} \otimes T_{n_3}^* + T_{n_3} \otimes \text{Ann}(V_{n_2}^*)$, the last two isomorphisms hold. \square

Theorem 4.6. *Let $n = n_1 < n_2 < \dots < n_i < \dots$ be the sequence of infinite length starting at n for $0 \leq n \leq \ell - 2$.*

- (1) Λ_n is rigid and indecomposable as a $U_q \times U_q$ -module. The Loewy series is given by

$$0 \subset \oplus_{i \geq 1} \mathbb{M}(L_{n_i}) \subset \sum_{i \geq 1} \mathbb{M}(V_{n_i}) + \sum_{i \geq 1} \mathbb{M}(V_{n_i}^*) \subset \sum_{i \geq 1} \mathbb{M}(T_{n_i}) = \Lambda_n$$

with layers $\oplus_{i \geq 1} L_{n_i} \otimes L_{n_i}$, $\oplus_{i \geq 1} (L_{n_{i+1}} \otimes L_{n_i} \oplus L_{n_i} \otimes L_{n_{i+1}})$ and $\oplus_{i \geq 1} L_{n_i} \otimes L_{n_i}$.

- (2) Λ_n also admits an increasing filtration of $U_q \times U_q$ -submodules

$$0 = P^0 \subset P^1 \subset \dots \subset P^i \subset \dots$$

and a decreasing filtration of $U_q \times U_q$ -submodules

$$\dots \subset Q^i \subset \dots \subset Q^2 \subset Q^1 \subset Q^0 = \Lambda_n$$

such that $\cup_i P^i = \Lambda_n$, $P^i/P^{i-1} \cong V_{n_i}^* \otimes V_{n_i}^*$, and $\cap_i Q^i = 0$, $Q^{i-1}/Q^i \cong V_{n_i} \otimes V_{n_i}$.

Proof. It follows from the three lemmas. For (2), set $P^i = \sum_{j \leq i} \mathbb{M}(T_{n_j})$ and $Q^i = \sum_{j \geq i+2} \mathbb{M}(T_{n_j})$. \square

Finally let's find out explicitly the cocommutative elements of O_q . Let $Y_n \subset O_q$ be the linear span of monomials $a^m b^k c^h, b^k c^h d^l$ of degree $\leq n$, i.e. $Y_n = \sum_{i \leq n} \mathbb{M}(T_i) = \sum_{i \leq n} \mathbb{M}(T_1)^i$.

Lemma 4.7. $Y_0 \subset Y_1 \subset \dots \subset Y_{n-1} \subset Y_n \subset \dots$ is a filtration of $U_q \times U_q$ -submodules of O_q with subquotients $Y_n/Y_{n-1} \cong V_n^* \otimes V_n^*$.

Proof. It follows from Lemma 4.4 (2) and Lemma 4.5 (2). \square

Lemma 4.8. The subspace $\{x \in V_n^* \otimes V_n^* : \rho'_1(u)x = \rho'_2(S^{-1}u)x, \forall u \in U_q\}$ is one-dimensional where ρ'_1, ρ'_2 denote the actions of U_q on the two copies of V_n^* .

Proof. Of course it follows from Lemma 3.4, the general version of it. But here we can compute more explicitly, which is actually the motivation behind the proof of Lemma 3.4.

Recall that V_n^* is $(n+1)$ -dimensional and has a basis e_0, e_1, \dots, e_n such that e_i is of weight $n-2i$ and $E^{(j)}e_i = \begin{bmatrix} i \\ j \end{bmatrix}_q e_{i-j}$; $F^{(j)}e_i = \begin{bmatrix} n-i \\ j \end{bmatrix}_q e_{i+j}$. For any $x = \sum_{i,j} x_{ij} e_i \otimes e_j \in V_n^* \otimes V_n^*$, if it satisfies that $\rho'_1(u^0)x = \rho'_2(S^{-1}u^0)x$ for any $u^0 \in U_q^0$, we must have $x_{ij} = 0$ except for $i+j = n$. Now let $x = \sum_i x_{i,n-i} e_i \otimes e_{n-i}$, since $S^{-1}E^{(j)} = (-1)^j q^{-j(j-1)} E^{(j)} K^{-j}$, it follows that

$$\rho'_1(E^{(j)})x = \sum_{i=j}^n x_{i,n-i} \begin{bmatrix} i \\ j \end{bmatrix}_q e_{i-j} \otimes e_{n-i}$$

and

$$\rho'_2(S^{-1}E^{(j)})x = \sum_{i=0}^{n-j} x_{i,n-i} (-1)^j q^{-j(j-1)} q^{-j(2i-n)} \begin{bmatrix} n-i \\ j \end{bmatrix}_q e_i \otimes e_{n-i-j}$$

for any $j \in \mathbb{N}, j \leq n$. Hence if $\rho'_1(E^{(j)})x = \rho'_2(S^{-1}E^{(j)})x$, then $x_{i,n-i} \begin{bmatrix} i \\ j \end{bmatrix}_q = x_{i-j,n-i+j}(-1)^j q^{j(j+n-2i+1)} \begin{bmatrix} n-i+j \\ j \end{bmatrix}_q$, in particular $x_{i,n-i} = x_{0,n}(-1)^i q^{i(n-i+1)} \begin{bmatrix} n \\ i \end{bmatrix}_q$, which implies that $x = x_{0,n}y$ with $y = \sum_{i=0}^n (-1)^i q^{i(n-i+1)} \begin{bmatrix} n \\ i \end{bmatrix}_q e_i \otimes e_{n-i}$. On the other hand it is straightforward to check that $\rho'_1(u)y = \rho'_2(S^{-1}u)y$ holds for any $u \in U_q$. \square

Proposition 4.9. $HH^0(O_q, O_q) = \mathbb{Q}(q)[a + d]$.

Proof. Denote the set of cocommutative elements of O_q by \mathbf{Co} . We need to show that \mathbf{Co} consists of polynomials in $a + d$. Recall that $O_q = \cup_{n \geq 0} Y_n$, and it is trivial that $Y_0 \cap \mathbf{Co} = \mathbb{Q}(q)$. Assume now that $Y_n \cap \mathbf{Co}$ is linearly spanned by polynomials of degree $\leq n$ in $a + d$. Suppose $f \in Y_{n+1} \cap \mathbf{Co}$, then $\rho_1(u)f = \rho_2(S^{-1}u)f$ for any $u \in U_q$. Since the image of $(a + d)^{n+1}$ in Y_{n+1}/Y_n is nonzero, by Lemma 4.8 there exists a scalar ζ such that $f - \zeta(a + d)^{n+1} \in Y_n$. Note that $f - \zeta(a + d)^{n+1}$ is also cocommutative, i.e. it belongs to $Y_n \cap \mathbf{Co}$, by induction $f - \zeta(a + d)^{n+1}$ is a polynomial of degree $\leq n$ in $a + d$, therefore f is a polynomial of degree $\leq n + 1$ in $a + d$. \square

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